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NOTE ON THE PRESSURE WITHIN THE EARTH.

It is the object of the present paper to briefly consider the magnitude of the pressures within the earth-spheroid, especially as influenced by the changes that have been brought about in the ellipticity of the earth's figure by its changing rotation period.

Darwin, in considering the stability of the moon-earth couple, says it seems improbable that a rotation of the earth in a little over five hours, with an ellipticity of $\frac{1}{12}$, would render the system unstable, and it hardly seems likely that better data and more perfect solution would largely affect the result, so as to make the period of revolution of the two bodies in the initial configuration very much less than five hours.[1] If the earth be assumed homogeneous throughout, as was done by Darwin in his investigations, with a density equal to the present mean density, it is a simple matter to calculate the pressures within the earth for any given eccentricity of its outer crust; and these eccentricities are, in turn, easily deducible from a knowledge of the rotation period. A table on page 327 of Part II of Thompson and Tait's Natural Philosophy gives us at once the rotation periods corresponding to various values of the eccentricity. We there find that

$e = .5$ corresponds to a rotation period of 15,730 seconds or $4\frac{1}{2}$ hours.
$e = .4$ corresponds to a rotation period of 19,780 seconds or $5\frac{1}{2}$ hours.

I have assumed that the separation of the moon-earth couple took place at a time when the rotation period of the earth was intermediate to the values just given, and that it would be sufficient for the purposes of geology to trace, from the epoch indicated, the changes in pressure that have taken place in the earth's interior. If it be assumed that the spheroids of eccen-

[1] Phil. Trans., 1879, Part 2, p. 536.

tricities .5 and .4 had the same volume and mass as the present earth, the polar and equatorial axes can readily be computed. Using Clark's value of the mean radius and volume, 6.3709×10^8 cm. (3958.8 mi.) and 1.0832×10^{27} cc. respectively, and Baily's value of the mean density, 5.67, I obtain the constants as given in lines 5–8 of Table I. The change of shape from the spherical

TABLE I.

		Spheroid 1	Spheroid 2	Sphere	Units
1	Eccentricity $= e$.	.5	.4	0	
2	Ellipticity $= \epsilon$ ...	.134	.0835	0	
3	Mean radius $= a_0$	6.3709×10^8 cc.	$=$ 3958.8 mi.		
4	Volume	1.0832×10^{27} cm.	$= 2.5988 \times 10^{11}$ cu. mi.		
5	Surface.........	197,800,000	197,160,000	196,950,000	sq. miles
6	Excess of surface over that of sphere........	850,000	210,000		sq. miles
7	Semi-polar axis..	3597	3736	3959	miles
8	Semi-equatorial axis..........	4155	4076	3959	miles
9	Attraction at pole	995.6	990.2	981	dynes
10	Attraction at equator.......	968.8	975.0	981	dynes
11	Rotation period if homogeneous..	15730	19780		seconds
12	Centripetal acceleration at equator...........	106.7	66.16		dynes
13	Gravity at equator...........	862.1	908.8		dynes
14	Pressure at center if homogeneous	1.633	1.688	1.772	million atmospheres
15	Ratio to pressure at center of sphere........	92.2	95.2		per cent.
16	Pressure at center if heterogeneous	2.88	2.92	3.00	million atmospheres
17	Ratio to pressure at center of sphere	96	97.5		per cent.
18	Change of volume, Laplacian law...........	2.1	1.3		per cent.
19	Percentage change in area, Laplacian law.	1.34	8.5		per cent.
20	Actual change in area, Laplacian law...........	2,700,000	1,700,000		sq. miles

form requires, of course, a change in area of surface, which change is noted in lines 5 and 6 of this table. The change in shape of the spheroid would likewise change the values of the attraction of gravitation at all points of the surface. The value of the attraction at the poles would be greater than the mean attraction on the surface of the present earth, while the attraction at the equator would be less. These values are placed in lines 9 and 10 of Table I. The determination of the attraction has been made in terms of the eccentricity from accurate formulas.[1] The values could have been computed in terms of the ellipticity[2] from the following approximate formulas, in which the square of the ellipticity has been neglected:

$$\text{attraction at pole} = (1 + \tfrac{2}{15}\,\epsilon)\,g_o.$$
$$\text{attraction at equator} = (1 - \tfrac{1}{15}\,\epsilon)\,g_o.$$

Here g_o is the attraction at the surface of the same mass in spherical form.

It should be noted in this connection that the ellipticities of the spheroids under consideration are so large as to render the omission of their squares unsafe, although, as is the case in the present paper, no great importance is to be attached to the actual figures of the results. For a like reason, Clairaut's theorem may not be used with much accuracy in checking results.

Besides the reduction in the attraction at the equator due to the change in the shape of the earth, there was formerly a still further loss due to the high centripetal acceleration accompanying the short rotation period. In the case of $e = .5$, this amounted to 107 dynes, and in the case of $e = .4$ to 66 dynes; these values subtracted from the values of the attraction previously determined, give the value of equatorial gravity placed in line 13 of Table I.

The values of gravity and pressure at any point on the polar or equatorial axis of the spheroid may now be determined. If

[1] See PRATT'S Figure of the Earth, 4th ed., p. 98.

[2] The ellipticity is the difference between major and minor axes divided by the major axis. I have represented it by the Greek ϵ, and have represented the eccentricity by e.

X_x represents the value of gravity at any point distant x from the center of the spheroid on an equatorial radius, and if Y_y represents the corresponding quantity for a point on the polar axis, and if g_e and g_p are the values of gravity at the equator and at the pole respectively, and if a and b are the semi-polar and semi-equatorial axes, then we have

$$X_x = \frac{g_e\, x}{a}\,; \qquad Y_y = \frac{g_p\, y}{b}\,.$$

Also if P_x and P_y represent the pressure at the same points, then

$$P_x = \frac{\rho_0}{2}\, g_e \left(a - \frac{x^2}{a}\right),$$

$$P_y = \frac{\rho_0}{2}\, g_p \left(b - \frac{y^2}{b}\right),$$

in which ρ_0 is the density of the homogeneous spheroid, and in which the other letters have the same significance as above. The following table gives the pressures at various distances from the center. The unit pressure is a million atmospheres of 10^6 dynes per sq. cm.

TABLE II.

PRESSURES WITHIN HOMOGENEOUS SPHEROIDS OF VARIOUS ECCENTRICITIES.

Distance from center along polar or equ. axis	$e = .5$	$e = .4$	$e = 0$	Distance from center along polar or equ. axis	$e = .5$	$e = .4$	$e = 0$
0	1.633	1.688	1.772	.6	1.043	1.079	1.133
.1	1.615	1.669	1.754	.7	.833	.862	.904
.2	1.567	1.618	1.700	.8	.588	.608	.638
.3	1.485	1.533	1.613	.9	.310	.321	.337
.4	1.370	1.417	1.488	1.0	.0	.0	.0
.5	1.224	1.264	1.328				

The pressures for $e = .5$ and for the sphere are shown graphically by the lower curves in Fig. 2. The line OX corresponds to either the polar or equatorial radius, as we may be pleased to consider it, but is represented, of course, as of length 10 in each case. The pressures at any other point in the spheroid can be found by drawing the equipotential surfaces; for on each of these the pressure is everywhere constant and equal, of course,

to the value of the pressure at the intersection of the equipotential surface with the polar and equatorial radii.

The pressures for $e = .4$ are not shown in the diagram, but they are not greatly different from those shown for $e = .5$. It should be noticed that the pressures for $e = .5$ are about 8 per cent. less than for the spherical form, and for $e = .4$ the pressures are about 5 per cent. less than for the spherical form.

The results above given were worked out on the supposition that the spheroid was homogeneous, having its density equal to the mean density of the earth. Of course the actual spheroid is not homogeneous, but heterogeneous, with the density increasing from surface to center. We know that the density of the surface material of the earth is approximately 2.75, and that the mean density is about twice as great. The exact law of variation of density in the interior cannot be said to be known, yet the law assigned by Laplace nearly a century ago is generally accepted as close to the truth. This law of density is as follows:

$$\rho = \frac{4.365\, a_0}{a} \sin \frac{2.4605\, a}{a_0};$$

in which ρ is the density of the stratum whose mean radius is a, the mean radius of the surface being a_0. The numerical constants are determined on the supposition that the surface density is 2.75 and the mean density twice as great. The variation according to this law is shown graphically by the heavily drawn curve of Fig. 3. An inspection of this diagram shows that the density increases quite uniformly for a considerable distance as we pass from the surface towards the center. We finally come to a central nucleus of nearly uniform density. The density at the center is 10:74.

The Laplacian law of density agrees well with the measurements of precession, and is probably as near to the truth as the measured values of the earth's mean density.

An exact method for determining the pressures within a heterogeneous spheroid without knowing its rotation period is not known to me. Even if the rotation period of the heteroge-

neous spheroids of eccentricities .5 and .4 were known, a computation of pressures would require the neglect of the squares of the ellipticities, which, in the case of ellipticities so large, would give results poorly compensating for the labor involved. I have, therefore, contented myself with two rough processes of approximation.

The pressure within a sphere in which the density is that of the Laplacian law can readily be computed by direct integration.[1] The result may be expressed as follows:

$$p = g_0 \, [2.7388] \left(\frac{\sin^2 n \, (2.4605)}{n^2} - 0.396 \right) \text{ atmospheres.}$$

In this formula, p is the pressure in atmospheres of 10^6 dynes per square cm. each, at the fractional distance n from the center of the earth, the radius being taken equal to unity for convenience. The bracket [2.7388] indicates the logarithm of a factor, and g_0 is the value of gravity at the surface.

Returning now to Fig. 1, it will be noticed that I have represented a section of the spheroid and two spheres, in contact at N. We shall suppose that the spheroid A is heterogeneous, with Laplace's law of density, and that sphere B is a sphere of same volume, same density, and the same law of density as the spheroid A. The sphere C is inscribed within the spheroid A, and has, I shall suppose, the same mean density and the same law of density as the latter. Then it is easy to see, since the law of density is such that the density increases towards the center, that the pressure at the center O of the sphere C must be less than the pressure at the center O of the spheroid A. Likewise, for the same law of density, the pressure at the point O of the sphere B is greater than the pressure at the center O of the spheroid A. The pressures at the point O within the spheres can be obtained by the formula above given, and as the pressure at the center of the spheroid is intermediate in value to those thus obtained, its value becomes approximately determined.

[1] See Osmond Fisher, Physics of the Earth's Crust, p. 32.

The pressures for the earth, calling the radius 10, are as follows, the unit being a million atmospheres:

Distance from center		Pressure in million atmospheres	Distance from center		Pressure in million atmospheres
0	- - -	3.00	6	- - -	1.25
1	- - - -	2.94	7	- - - -	0.85
2	- - -	2.74	8	- - -	0.50
3	- - - -	2.46	9	- - - -	0.21
4	- - -	2.09	10	- - -	0.00
5	- - - -	1.67			

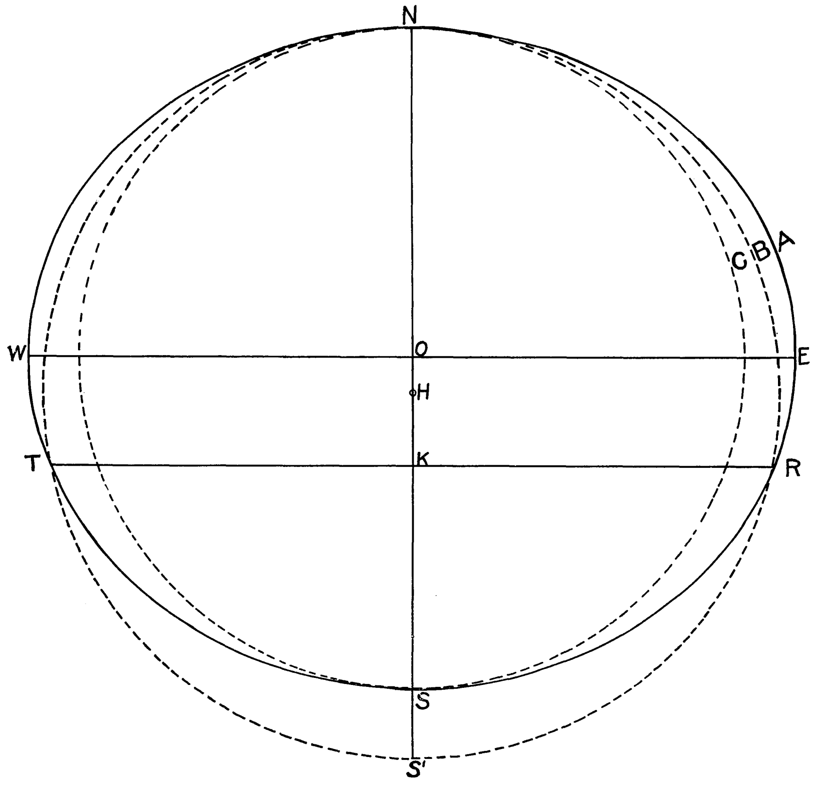

FIG. 1.

The pressures in the spheroid if $e=.5$ are about 4 per cent less, and if $e=.4$ are about $2\frac{1}{2}$ per cent. less than pressures

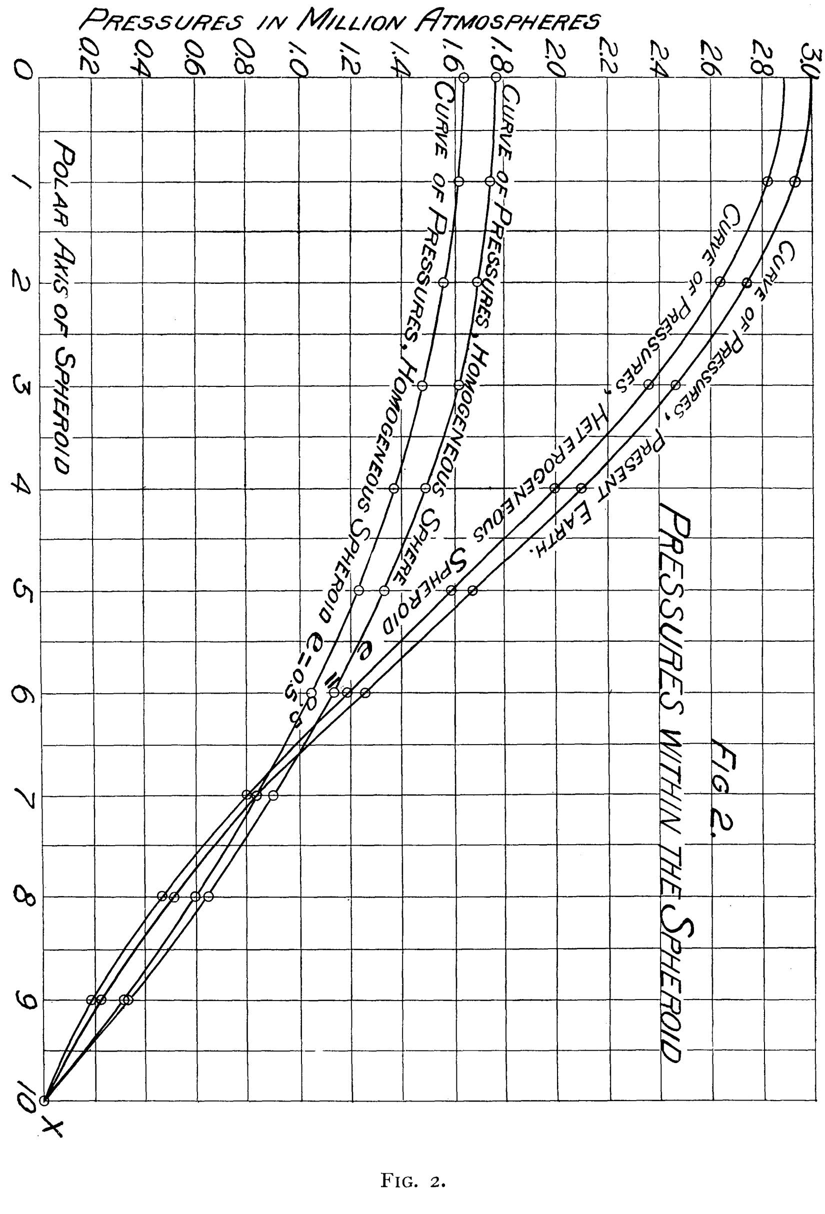
Pressures in Million Atmospheres
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
0
1
2
3
4
5
6
7
8
9
10
X
Polar Axis of Spheroid
Curve of Pressures, Homogeneous Spheroid e=0.5
Curve of Pressures, Homogeneous Sphere e = 0
Curve of Pressures, Heterogeneous Spheroid
Curve of Pressures, Present Earth.
Pressures within the Spheroid
Fig 2.

Fig. 2.

in present earth. The pressures in the present earth and in the heterogeneous spheroid if $e=.5$ are shown graphically by the upper curves in Fig. 2. The greatest rate of change of pressures is seen to be at a point about .65 of the distance from the center to the surface.

Whatever the law of increase of density within the spheroid, provided only that the density continually increases as we approach the center, we may easily derive the following theorem:

The pressure at the center of a heterogeneous spheroid differs from the pressure at the center of the same matter in the spherical form, by a fractional amount which is less than two-thirds the ellipticity of the spheroid.

Thus if the ellipticity is .06, the pressure at the center will be not to exceed 4 per cent. less than if the matter was in the spherical form. This shows that the changes in pressure due to the changing ellipticity of the earth are limited in amount, although important, and of the same order of magnitude as the ellipticity.

Another roughly approximate method of estimating the pressures within the heterogeneous spheroid consists in assuming that all the strata of equal density have the same ellipticity as the surface. As a matter of fact, the ellipticity of the strata decrease as we approach the center by a law which may be deduced from the Laplacian law of density, and which is represented graphically by the broken line in Fig. 3. The ordinate of this curve gives the ratio of the ellipticity of a stratum to the ellipticity of the surface. It will be observed that the ellipticity of strata near the center is about 80.72 per cent. of the surface value. The actual case, then, does not differ from the assumed case of uniform ellipticity by a very large amount. It leads to the result that the change in pressure at the center of the earth due to a change in the ellipticity of the outer crust, is nearly the same in amount as if the earth were homogeneous, although the percentage change is much less than in the latter case. The relation between the ellipticity of any stratum to surface ellipticity is given by the equation :

$$\epsilon = \frac{2}{a^2}\left(1 - \frac{q^2\, a^2}{3\,(1 - q\, a \cot q\, a)} \right) \epsilon_0$$

in which ϵ is the ellipticity of the stratum whose mean radius is a, ϵ_0 is the ellipticity of the surface, and $q = \frac{2.4605}{a_0}$[1]

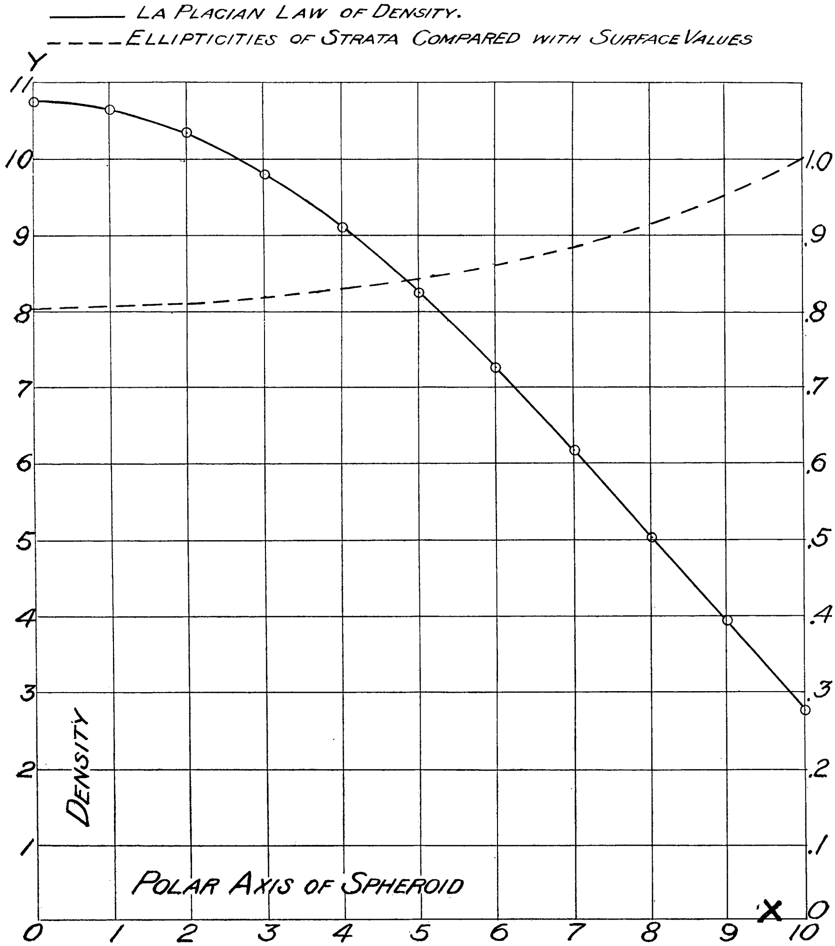

FIG. 3.

[1] See Thompson and Tait, II, p. 410; Pratt, Figure of the Earth, 4th ed., p. 115; Clarke, Geodosy, p. 84.

The decrease in the size of the earth that would be brought about by the increase of internal pressure discussed above, may be computed, if it be assumed that the high density in the interior of the earth is due alone to the compressibility of matter under the enormous pressures there present. Laplace, as a matter of fact, began by assigning a law of compressibility and thence deducing the law of density; however, his law of density might be close to the truth and yet be not entirely controlled by the pressure. The equation

$$\rho = 4.365 \frac{a_0}{a} \sin \frac{2.4605\, a}{a_0}$$

leads to the following relation connecting pressure and density.[1]

$$d\,p = k\, d\,(\rho^2)$$

or, as stated by Laplace: *The variation in pressure in the interior of the earth is proportional to the variation in the square of the density.*

The law of compressibility of gases — "Boyle's Law" — and the law of elasticity for small compressions in solids — "Hooke's Law" — state that the variation in density is directly proportional to the variation in pressure. Thus Laplace's law assumes a compressibility which is less than that given by either of the laws just named, an assumption which is, in itself, very reasonable.

From the above equation we derive

$$k\,\rho^2 = p + c$$

and determining the constants on the supposition that the surface density is 2.75, and the central density 10.74, we conclude that

$$(.02575)\,\rho^2 = p + .1947$$

in which p must be given in terms of a million atmospheres as unit. To determine the change in density due to a small change in pressure we may write,

$$\frac{d\,p}{p+c} = 2\,\frac{d\,\rho}{\rho},$$

or

$$\frac{d\,p}{p}\left(\frac{p}{p+c}\right) = 2\,\frac{d\,\rho}{\rho}.$$

[1] See O'Brien, Math. Tracts, p. 39; Pratt, p. 113; Thompson and Tait, II, p. 403.

Now if the change in pressure that has taken place is 4 per cent., we may place

$$\frac{dp}{p} = \frac{4}{100},$$

and, since c is small in comparison with p,

$$\frac{d\rho}{\rho} = \tfrac{1}{2}\,\frac{dp}{p} - n = \frac{2}{100} - n$$

in which n is a number with a small average value. Therefore, we may assume that, on the average, the change in density is about half the change in pressure. Taking the ratio as one-half, we can compute the change in volume of the spheroid for a given change in internal pressure, since, of course, volume is inversely proportional to density. I have placed results of this computation in the line 18 of Table I.

The decrease in the size of the earth due to the increase of internal pressures must likwise reduce the extent of the outer surface. If v and s represent the volume and surface of the sphere,

$$\frac{ds}{s} = \tfrac{2}{3}\,\frac{dv}{v},$$

or, for small changes, the change in surface is two-thirds the change in volume. This gives a reduction in surface amounting to about 1.3 per cent. for the spheroid $e = .5$, and .85 per cent. for the spheroid $e = .4$, or, in square miles, a reduction in surface of about 2,700,000 square miles, and 1,700,000 square miles respectively.

The compressibility of matter under high pressure and high temperature cannot be said to be known experimentally. Thompson and Tait, however, on page 415 of Part II, estimate for the average material at the surface of the earth, the compressibility that must theoretically follow from Laplace's law, and give as the result:

Melted lava, by Laplace's law - - 4.42

They also give actual experimental determinations of compressibility as follows:[1]

[1] If the radius of the earth, 6.37×10^8, be divided by each of these numbers and if

Alcohol	37.
Water	29.
Mercury	27.
Glass	5.0
Copper	8.1
Iron	4.1

The comparison, they remark, may well be considered as decidedly not adverse to Laplace's law. We may thus infer that it is far from unreasonable to hold that the high density of the interior of the earth is entirely due to the enormous pressures there present and hence it is not unreasonable to hold that the diminution in the earth's surface, pointed out above, has actually taken place. Yet, whatever future experiments may show in regard to the compressibility of melted rock, we certainly must believe that the enormous pressure at the center of the earth does have some effect, and, indeed, a large effect, in making the density high in that part of the interior.

In addition to the results which must follow from a former high rotation period of the earth and large ellipticity, important increments to the internal pressures must have taken place, if any change in the interior from homogeneity to heterogeneity has occurred. Notwithstanding the current computed results for the cooling of the earth, it seems reasonable to suppose that the energy in the interior of the earth, was, within geological times, distributed with greater homogeneity than it is at present. If any such change in the distribution of energy has taken place, then the density of the earth's interior has likewise progressed from homogeneity to heterogeneity. The curves given in Fig. 2 show that the pressure at the center of a homogeneous spheroid is only about half the pressure at the center of the present earth. Therefore, any progress that has been made

each quotient thus obtained be multiplied by 981 times the density of the substance, the result will be the *volume elasticity* in dynes per sq. cm. If the reciprocal of this last result be multiplied by 10^6, the result will be the *compression per atmosphere*. The numbers given in the table divided into the radius of the earth give what Thompson and Tait call the "lengths of the moduli of compression." See THOMPSON and TAIT, II, p. 225, § 689.

from homogeneity to heterogeneity would result in increased pressure in the interior, and in a decreased magnitude of the earth's volume and surface.

It is difficult to decide whether or not the minimum value of the eccentricity used above is too high to correspond with that uncertain date "the beginning of geological time." The rotation period of the heterogeneous spheroid would, probably, be shorter than for the homogeneous spheroid, and the shorter period may not be consistent with theories regarding the moon-earth couple. If we are required to assume a value of the eccentricity less than that used above, the changes in pressures I have given must be reduced. On the other hand, it must be remembered that there are causes at work which may augment the effects of a change in the internal pressure, and may even produce large results from what seem to be small causes. For example, if we suppose a contest in a given region in the interior between extreme heat on the one hand, and extreme pressure on the other hand, as to whether the material, or a single constituent of the material, will take on the crystalline form or not, we have a case in point. It may happen that a very slight increase in pressure may materially extend the zone in which crystallization may take place and thus result in a considerable increase in density; it is not impossible to believe that such a zone may exist in the region near the center, where pressures may be dominant on account of their enormous magnitude, and also in a region near the surface, where pressure may again be dominant owing to the lower temperature.

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